

Number of points of a nonsingular hypersurface in an odd-dimensional projective space

Masaaki Homma ^{*}

Department of Mathematics and Physics
Kanagawa University
Hiratsuka 259-1293, Japan
homma@kanagawa-u.ac.jp

Seon Jeong Kim [†]

Department of Mathematics and RINS
Gyeongsang National University
Jinju 660-701, Korea
skim@gnu.kr

Abstract

The numbers of \mathbb{F}_q -points of nonsingular hypersurfaces of a fixed degree in an odd-dimensional projective space are investigated, and an upper bound for them is given. Also we give the complete list of nonsingular hypersurfaces each of which realizes the upper bound. This is a natural generalization of our previous study of surfaces in projective 3-space.

Key Words: Finite field, Hypersurface, Hermitian variety

MSC: 14G15, 14N05, 14J70

1 Introduction

Several years ago, we established the elementary bound for the numbers of \mathbb{F}_q -points of hypersurfaces of projective n -space \mathbb{P}^n with $n \geq 3$ [3], and later gave the complete list of surfaces in \mathbb{P}^3 whose number of \mathbb{F}_q -points reached this bound [4, 5]. Recently Tironi extended this list for hypersurfaces in \mathbb{P}^n [10]. Although surfaces appeared in the list are nonsingular, hypersurfaces appeared in the extended list with $n > 3$ are cones over those surfaces except when the degree of the hypersurface is $q + 1$. Therefore if we restrict our investigation within nonsingular hypersurfaces, we can expect a tighter bound than the elementary bound.

^{*}Partially supported by JSPS KAKENHI Grant Number JP15K04829.

[†]Partially supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2016R1D1A1B01011730).

We fix a finite field \mathbb{F}_q of q elements. The number of \mathbb{F}_q -points of the projective m -space is denoted by $\theta_q(m)$, that is, $\theta_q(m) = \sum_{\nu=0}^m q^\nu$. A closed subscheme \mathcal{X} in \mathbb{P}^m over the algebraic closure of \mathbb{F}_q is said to be defined over \mathbb{F}_q if the homogeneous ideal of \mathcal{X} is generated by polynomials $f_1(X_0, \dots, X_m), \dots, f_s(X_0, \dots, X_m)$ in $\mathbb{F}_q[X_0, \dots, X_m]$. An \mathbb{F}_q -point (a_0, \dots, a_m) of \mathbb{P}^m is said to be an \mathbb{F}_q -point of \mathcal{X} if $f_1(a_0, \dots, a_m) = \dots = f_s(a_0, \dots, a_m) = 0$, namely we do not care the point is a multiple point or not in \mathcal{X} . The set of \mathbb{F}_q -points of \mathcal{X} is denoted by $\mathcal{X}(\mathbb{F}_q)$ and the cardinality of this set by $|\mathcal{X}(\mathbb{F}_q)|$ or $N_q(\mathcal{X})$. We frequently use the notation $\{f_1 = \dots = f_m = 0\}$ for the scheme \mathcal{X} .

Geometric structure of \mathcal{X} , for example, nonsingularity or irreducibility, is normally (and also in this article) considered over the algebraic closure $\overline{\mathbb{F}}_q$ of \mathbb{F}_q , but we are just interesting in the set-theoretical counting of $\mathcal{X}(\mathbb{F}_q)$.

The purpose of this article is to show the following theorem.

Theorem 1.1 *Let n be an odd integer at least 3. If X is a nonsingular hypersurface of degree $d \geq 2$ in \mathbb{P}^n defined over \mathbb{F}_q . Then*

$$N_q(X) \leq \theta_q\left(\frac{n-1}{2}\right) \cdot \left((d-1)q^{\frac{n-1}{2}} + 1\right),$$

and equality holds if and only if either

- (i) $d = 2$ and X is the nonsingular hyperbolic quadric hypersurface, that is, X is projectively equivalent over \mathbb{F}_q to the hypersurface

$$\sum_{i=0}^{\frac{n-1}{2}} X_{2i} X_{2i+1} = 0; \text{ or}$$

- (ii) $d = \sqrt{q} + 1$ where q is square, and X is a nonsingular Hermitian hypersurface, that is, X is projectively equivalent over \mathbb{F}_q to the hypersurface

$$\sum_{i=0}^{\frac{n-1}{2}} \left(X_{2i}^{\sqrt{q}} X_{2i+1} + X_{2i} X_{2i+1}^{\sqrt{q}} \right) = 0; \text{ or}$$

- (iii) $d = q + 1$ and X is a nonsingular \mathbb{P}^n -filling hypersurface over \mathbb{F}_q , that is, X is projectively equivalent over \mathbb{F}_q to the hypersurface

$$\sum_{i=0}^{\frac{n-1}{2}} (X_{2i}^q X_{2i+1} - X_{2i} X_{2i+1}^q) = 0.$$

We prove this by induction on n , so $n = 3$ is the first step of the induction, which was already showed in [4, Theorem 1]:

Theorem 1.2 *Let X be a surface of degree d in \mathbb{P}^3 over \mathbb{F}_q without \mathbb{F}_q -plane components. Then $N_q(X) \leq \theta_q(1) \cdot ((d-1)q+1)$, and equality holds if and only if the degree d is either 2 or $\sqrt{q}+1$ (when q is a square) or $q+1$ and the surface X is projectively equivalent to one of the following surfaces over \mathbb{F}_q according to the degree:*

- (i) $X_0X_1 + X_2X_3 = 0$ if $d = 2$;
- (ii) $X_0^{\sqrt{q}}X_1 + X_0X_1^{\sqrt{q}} + X_2^{\sqrt{q}}X_3 + X_2X_3^{\sqrt{q}} = 0$ if $d = \sqrt{q} + 1$;
- (iii) $X_0^qX_1 - X_0X_1^q + X_2^qX_3 - X_2X_3^q = 0$ if $d = q + 1$.

Remark 1.3 (i) The assumption that X has no \mathbb{F}_q -plane components in the above theorem is milder than the nonsingularity of X if $\deg X \geq 2$.

- (ii) Equations in the above theorem and those in [4, Theorem 1] are seemingly different. But one can easily confirm that in each degree those equations are projectively equivalent over \mathbb{F}_q to each other.

2 Preliminary

This section is a mixture of facts that are mostly independent of one another, but necessary to our proof.

We keep roman letters X, Y, Z for particular varieties for later use. In this section, varieties or schemes are denoted by calligraphic letters $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ etc.

2.1 A necessary condition of a hypersurface to be nonsingular

Lemma 2.1 *Let \mathcal{X} be a hypersurface of degree ≥ 2 in \mathbb{P}^m over an algebraically closed field, and \mathcal{L} a linear subspace of \mathbb{P}^m which is contained in \mathcal{X} . If \mathcal{X} is nonsingular, then $\dim \mathcal{L} \leq \lfloor \frac{m-1}{2} \rfloor$. Here the symbol $\lfloor \frac{m-1}{2} \rfloor$ denotes the integer part of $\frac{m-1}{2}$.*

Proof. Let $r = \dim \mathcal{L}$. Choose the coordinates X_0, \dots, X_m of \mathbb{P}^m so that \mathcal{L} is defined by $X_0 = X_1 = \dots = X_{m-r-1} = 0$. Since $\mathcal{L} \subset \mathcal{X}$, the equation of \mathcal{X} is of the form

$$F(X_0, \dots, X_m) = \sum_{i=0}^{m-r-1} f_i(X_0, \dots, X_m) X_i = 0.$$

Note that each homogeneous polynomial $f_i(X_0, \dots, X_m)$ is not constant because $\deg \mathcal{X} \geq 2$. Consider the simultaneous equations

$$F = \frac{\partial F}{\partial X_0} = \dots = \frac{\partial F}{\partial X_m} = 0,$$

more explicitly:

$$\left\{ \begin{array}{l} F = \sum_{i=0}^{m-r-1} f_i X_i = 0 \\ \frac{\partial F}{\partial X_0} = \sum_{i=0}^{m-r-1} \frac{\partial f_i}{\partial X_0} X_i + f_0 = 0 \\ \vdots \\ \frac{\partial F}{\partial X_{m-r-1}} = \sum_{i=0}^{m-r-1} \frac{\partial f_i}{\partial X_{m-r-1}} X_i + f_{m-r-1} = 0 \\ \frac{\partial F}{\partial X_{m-r}} = \sum_{i=0}^{m-r-1} \frac{\partial f_i}{\partial X_{m-r}} X_i = 0 \\ \vdots \\ \frac{\partial F}{\partial X_m} = \sum_{i=0}^{m-r-1} \frac{\partial f_i}{\partial X_m} X_i = 0. \end{array} \right. \quad (1)$$

We may view $\{X_{m-r}, \dots, X_m\}$ as a system of coordinates of $\mathcal{L} = \mathbb{P}^r$. Suppose $m - r \leq r$. Then the simultaneous $m - r$ equations

$$\left\{ \begin{array}{l} f_0(0, \dots, 0, X_{m-r}, \dots, X_m) = 0 \\ \vdots \\ f_{m-r-1}(0, \dots, 0, X_{m-r}, \dots, X_m) = 0 \end{array} \right. \quad (2)$$

has a solution $(\alpha_{m-r}, \dots, \alpha_m)$ in \mathbb{P}^r . Hence the point $(0, \dots, 0, \alpha_{m-r}, \dots, \alpha_m)$ in $\mathcal{L} \subset \mathcal{X}$ is a solution of (1), which must be a singular point of \mathcal{X} . Therefore we have $m - r > r$ if \mathcal{X} is nonsingular. \square

2.2 Segre-Serre-Sørensen bound

Without any restrictions on a hypersurface over \mathbb{F}_q , the best bound was obtained by Serre [7], which is a generalization of Segre's old result for plane curves [6]. Sørensen [8] also proved the same inequality as Serre's.

Lemma 2.2 (Segre-Serre-Sørensen) *Let $\mathcal{X} \subset \mathbb{P}^m$ be a hypersurface of degree d defined over \mathbb{F}_q . Then $N_q(\mathcal{X}) \leq dq^{m-1} + \theta_q(m-2)$. Moreover, when “ $m = 2$ ” or “ $m > 2$ and $d \leq q$ ”, equality holds if and only if there are d hyperplanes $\mathcal{L}_1, \dots, \mathcal{L}_d$ over \mathbb{F}_q such that $\mathcal{X} = \cup_{i=1}^d \mathcal{L}_i$ and $\cap_{i=1}^d \mathcal{L}_i$ is of dimension $m - 2$.*

Proof. See [6, II §6 Observation IV] for “ $m = 2$ ”, [7] for “ $m > 2$ ”.

Notation 2.3 For a variety \mathcal{X} , $\text{Sing } \mathcal{X}$ denotes the locus of singular points.

In Lemma 2.2, $\text{Sing } \mathcal{X} = \bigcap_{i=1}^d \mathcal{L}_i$. Actually, the following lemma holds.

Lemma 2.4 *Let \mathcal{X} be a hypersurface in \mathbb{P}^m . If \mathcal{X} splits into hyperplanes : $\mathcal{X} = \bigcup_{i=1}^d \mathcal{L}_i$, then $\text{Sing } \mathcal{X} = \bigcup_{i < j} (\mathcal{L}_i \cap \mathcal{L}_j)$.*

Proof. Let $g_i = \sum_{j=0}^m a_{ij} X_j = 0$ be the linear equation of \mathcal{L}_i . So \mathcal{X} is defined by $G = \prod_{i=1}^d g_i = 0$. Then

$$\frac{\partial G}{\partial X_\nu} = \sum_{i=1}^d a_{i\nu} \prod_{\substack{l \text{ with} \\ l \neq i}} g_l.$$

If $(u_0, \dots, u_m) \in \mathcal{L}_\alpha \cap \mathcal{L}_\beta$, then $\frac{\partial G}{\partial X_\nu}(u_0, \dots, u_m) = 0$ because g_α or g_β appears in each term of $\frac{\partial G}{\partial X_\nu}$. Hence $\mathcal{L}_\alpha \cap \mathcal{L}_\beta \subset \text{Sing } \mathcal{X}$. Conversely if $(u_0, \dots, u_m) \in \mathcal{X} \setminus \bigcup_{i < j} (\mathcal{L}_i \cap \mathcal{L}_j)$, there is a unique hyperplane \mathcal{L}_ν which contains the point (u_0, \dots, u_m) . Hence $\frac{\partial G}{\partial X_\nu}(u_0, \dots, u_m) = a_{\alpha\nu} \prod_{l \neq \alpha} g_l(u_0, \dots, u_m)$, is nonzero for some ν . Hence $\mathcal{X} \setminus \bigcup_{i < j} (\mathcal{L}_i \cap \mathcal{L}_j) \subset \mathcal{X} \setminus \text{Sing } \mathcal{X}$. \square

We frequently use the latter half of Segre-Serre-Sørensen's lemma (2.2). For the convenience of readers, we reformulate the necessary part with a small generalization and give its proof.

Lemma 2.5 *Let $\mathcal{L}_1, \dots, \mathcal{L}_d$ be distinct linear subspaces over \mathbb{F}_q in \mathbb{P}^m such that*

(i) $\dim \mathcal{L}_1 = \dots = \dim \mathcal{L}_d = k$, and

(ii) $\dim \bigcap_{i=1}^d \mathcal{L}_i = k - 1$.

Then $N_q(\mathcal{L}_1 \cup \dots \cup \mathcal{L}_d) = dq^k + \theta_q(k - 1)$.

Proof. Let $\Lambda = \bigcap_{i=1}^d \mathcal{L}_i$. From the assumptions, $\mathcal{L}_i \cap \mathcal{L}_j = \Lambda$ if $i \neq j$. Therefore

$$(\mathcal{L}_1 \cup \dots \cup \mathcal{L}_d)(\mathbb{F}_q) = \left(\prod_{i=1}^d (\mathcal{L}_i(\mathbb{F}_q) \setminus \Lambda(\mathbb{F}_q)) \right) \coprod \Lambda(\mathbb{F}_q),$$

where the symbol \coprod means taking the disjoint union. Hence $N_q(\mathcal{L}_1 \cup \dots \cup \mathcal{L}_d) = d(\theta_q(k) - \theta_q(k - 1)) + \theta_q(k - 1)$. \square

The next lemma is also useful.

Lemma 2.6 *Let \mathcal{X} be a hypersurface of \mathbb{P}^m , and \mathcal{S} a linear subspace of \mathbb{P}^m such that $\mathcal{S} \not\subset \mathcal{X}$. If a point $Q \in \mathcal{S} \cap \mathcal{X}$ is nonsingular in $\mathcal{S} \cap \mathcal{X}$, then Q is also nonsingular in \mathcal{X} .*

Proof. We assume that $\mathcal{S} = \{X_0 = \cdots = X_s = 0\}$ and $Q = (0, \dots, 0, 1)$. Use affine coordinates $x_0 = \frac{X_0}{X_m}, \dots, x_{m-1} = \frac{X_{m-1}}{X_m}$. Let $f(x_0, \dots, x_{m-1}) = f_1 + f_2 + \cdots + f_d = 0$ be the local equation of \mathcal{X} around Q , where $f_i = f_i(x_0, \dots, x_{m-1})$ is the homogeneous part of degree i of f . Since Q is nonsingular in $\mathcal{S} \cap \mathcal{X}$, $f_1(0, \dots, 0, x_{s+1}, \dots, x_{m-1})$ is nontrivial. Hence so is $f_1(x_1, \dots, x_{m-1})$. \square

2.3 Cone lemma

Lemma 2.7 *Let $f(X_0, \dots, X_m)$ be a homogeneous polynomial over \mathbb{F}_q of degree $d \leq q$. If $f(a_0, \dots, a_m) = 0$ for any $(a_0, \dots, a_m) \in \mathbb{F}_q^n$, then f is the zero polynomial.*

Proof. Suppose f is nontrivial, then it defines a hypersurface \mathcal{X} of degree d in \mathbb{P}^m . By the lemma of Segre-Serre-Sørensen (2.2),

$$\begin{aligned} N_q(\mathcal{X}) &\leq dq^{m-1} + \theta_q(m-2) \\ &\leq q^m + \theta_q(m-2) < \theta_q(m) \quad \text{if } d \leq q, \end{aligned}$$

however, $\mathcal{X}(\mathbb{F}_q) = \mathbb{P}^m(\mathbb{F}_q)$ by the assumption; these contradict each other. Hence the polynomial must be trivial. \square

The following fact, which will be referred to as “cone lemma”, is a bridge between point-counting and geometry.

Proposition 2.8 *Let \mathcal{X} be a hypersurface over \mathbb{F}_q of degree d with $2 \leq d \leq q$ in \mathbb{P}^m , and $\mathcal{L} = \mathbb{P}^{m-k}$ an \mathbb{F}_q -linear subvariety of \mathbb{P}^m of dimension $m-k$, where $3 \leq k \leq m$. Let $\mathcal{M} = \mathbb{P}^{k-1}$ be another \mathbb{F}_q -linear subvariety of \mathbb{P}^m of dimension $k-1$ such that $\mathcal{L} \cap \mathcal{M} = \emptyset$, and \mathcal{Y} a hypersurface of degree d in $\mathbb{P}^{k-1} = \mathcal{M}$ over \mathbb{F}_q . Suppose that*

$$\begin{aligned} N_q(\mathcal{Y}) &> (d-1)q^{k-2} + \theta_q(k-3), \\ \mathcal{X} &\supset \mathcal{Y}, \text{ and} \\ \mathcal{X} &\supset (\mathbb{P}^{m-k} * \mathcal{Y})(\mathbb{F}_q), \end{aligned}$$

where $\mathbb{P}^{m-k} * \mathcal{Y}$ denotes the cone over \mathcal{Y} with center $\mathbb{P}^{m-k} = \mathcal{L}$. Then $\mathcal{X} = \mathbb{P}^{m-k} * \mathcal{Y}$.

Proof. Choose coordinates $X_0, \dots, X_{k-1}, X_k, \dots, X_m$ of \mathbb{P}^m so that $\mathcal{L} = \mathbb{P}^{m-k}$ is defined by $X_0 = \cdots = X_{k-1} = 0$, and $\mathcal{M} = \mathbb{P}^{k-1}$ by $X_k = \cdots = X_m = 0$. Let

$$F(X_0, \dots, X_m) = \sum_{\substack{e=(e_0, \dots, e_m) \\ \text{with} \\ e_0 + \cdots + e_m = d}} \alpha_e X_0^{e_0} \cdots X_m^{e_m} = 0$$

be the equation of \mathcal{X} . Note that the polynomial F can be rewritten as

$$\begin{aligned} F(X_0, \dots, X_m) = & \sum_{\mu=0}^d \sum_{\substack{(e_k, \dots, e_m) \\ \text{with} \\ \sum_k^m e_j = \mu}} \left(\sum_{\substack{(e_0, \dots, e_{k-1}) \\ \text{with} \\ \sum_0^{k-1} e_i = d - \mu}} \alpha_{(e_0, \dots, e_{k-1}, e_k, \dots, e_m)} X_0^{e_0} \cdots X_{k-1}^{e_{k-1}} \right) X_k^{e_k} \cdots X_m^{e_m}. \quad (3) \end{aligned}$$

Let $(0, \dots, 0, b_k, \dots, b_m) \in \mathcal{L}(\mathbb{F}_q)$ and $(a_0, \dots, a_{k-1}, 0, \dots, 0) \in \mathcal{Y}(\mathbb{F}_q)$. Since $\mathcal{X} \supset (\mathbb{P}^{m-k} * \mathcal{Y})(\mathbb{F}_q)$,

$$(ta_0, \dots, ta_{k-1}, sb_k, \dots, sb_m) \in \mathcal{X}(\mathbb{F}_q) \quad (4)$$

for any $(s, t) \in \mathbb{P}^1(\mathbb{F}_q)$. Substitute (4) for $F(X_0, \dots, X_m)$, then by (3)

$$\sum_{\mu=0}^d t^{d-\mu} s^\mu \sum_{\substack{(e_k, \dots, e_m) \\ \text{with} \\ \sum_k^m e_j = \mu}} \left(\sum_{\substack{(e_0, \dots, e_{k-1}) \\ \text{with} \\ \sum_0^{k-1} e_i = d-\mu}} \alpha_{(e_0, \dots, e_{k-1}, e_k, \dots, e_m)} a_0^{e_0} \dots a_{k-1}^{e_{k-1}} \right) b_k^{e_k} \dots b_m^{e_m} = 0 \quad (5)$$

for any $(s, t) \in \mathbb{P}^1(\mathbb{F}_q)$. Since $d \leq q$ but $|\mathbb{P}^1(\mathbb{F}_q)| = q + 1$, all coefficients of the polynomial (5) in s and t are 0. Hence

$$\sum_{\substack{(e_k, \dots, e_m) \\ \text{with} \\ \sum_k^m e_j = \mu}} \left(\sum_{\substack{(e_0, \dots, e_{k-1}) \\ \text{with} \\ \sum_0^{k-1} e_i = d-\mu}} \alpha_{(e_0, \dots, e_{k-1}, e_k, \dots, e_m)} a_0^{e_0} \dots a_{k-1}^{e_{k-1}} \right) b_k^{e_k} \dots b_m^{e_m} = 0 \quad (6)$$

for any $(a_0, \dots, a_{k-1}, 0, \dots, 0) \in \mathcal{Y}(\mathbb{F}_q)$ and $(b_k, \dots, b_m) \in \mathbb{P}^{m-k}(\mathbb{F}_q)$.

First we fix the element $(a_0, \dots, a_{k-1}) \in \mathcal{Y}(\mathbb{F}_q)$ and view (6) as a polynomial with variables (b_k, \dots, b_m) . Since the degree of the polynomial (6) in (b_k, \dots, b_m) is μ ($\leq d \leq q$) and it is 0 for any $(b_k, \dots, b_m) \in \mathbb{P}^{m-k}(\mathbb{F}_q)$, it must be a zero polynomial by (2.7), that is

$$\sum_{\substack{\mathbf{e}=(e_0, \dots, e_{k-1}) \\ \text{with} \\ e_0 + \dots + e_{k-1} = d-\mu}} \alpha_{(\mathbf{e}, e_k, \dots, e_m)} a_0^{e_0} \dots a_{k-1}^{e_{k-1}} = 0$$

for any (e_k, \dots, e_m) with $e_k + \dots + e_m = \mu$. Hence for each $\mathbf{e}' = (e_k, \dots, e_m)$ the hypersurface $\mathcal{Y}_{\mathbf{e}'}$ defined by

$$\sum_{\substack{\mathbf{e}=(e_0, \dots, e_{k-1}) \\ \text{with} \\ e_0 + \dots + e_{k-1} = d-\mu}} \alpha_{(\mathbf{e}, \mathbf{e}')} X_0^{e_0} \dots X_{k-1}^{e_{k-1}} = 0$$

in $\mathbb{P}^{k-1} = \mathcal{M}$ contains $\mathcal{Y}(\mathbb{F}_q)$. If the polynomial is nontrivial (of degree $d - \mu$), then $N_q(\mathcal{Y}_{\mathbf{e}'}) \leq (d - \mu)q^{k-2} + \theta_q(k - 3)$ by Lemma 2.2. On the other hand, $N_q(\mathcal{Y}) > (d - 1)q^{k-2} + \theta_q(k - 3)$ by the assumption. Hence if $\mu \geq 1$, this polynomial must be trivial. Therefore \mathcal{X} is a cone of the hypersurface

$$\sum_{\substack{(e_0, \dots, e_{k-1}) \\ \text{with} \\ e_0 + \dots + e_{k-1} = d}} \alpha_{(e_0, \dots, e_{k-1}, 0, \dots, 0)} X_0^{e_0} \dots X_{k-1}^{e_{k-1}} = 0$$

of \mathbb{P}^{k-1} , which is the equation of \mathcal{Y}_0 . In particular, $\mathcal{X} \cap \mathbb{P}^{k-1} = \mathcal{Y}_0$ and $\mathcal{X} = \mathbb{P}^{m-k} * \mathcal{Y}_0$. Since $\mathcal{X} \cap \mathbb{P}^{k-1} \supset \mathcal{Y}$ by the assumption and $\deg \mathcal{X} = \deg \mathcal{Y}$, we have $\mathcal{Y}_0 = \mathcal{Y}$. This completes the proof. \square

3 A bound involving Koen Thas' invariant

In [9], Koen Thas defined an invariant of a hypersurface (see, Definition 3.1 below) and obtained a bound for $N_q(\mathcal{X})$'s, which involved the invariant. We now give a simpler bound than his. A comparison his bound and ours will give in the last section.

Definition 3.1 Let \mathcal{X} be a hypersurface defined over \mathbb{F}_q in \mathbb{P}^m . The maximum dimension of an \mathbb{F}_q -linear subspace of \mathbb{P}^m which is contained in \mathcal{X} is denoted by $k_{\mathcal{X}}$.

By Lemma 2.1, if \mathcal{X} is nonsingular and $\deg \mathcal{X} \geq 2$, then $k_{\mathcal{X}} \leq \lfloor \frac{m-1}{2} \rfloor$.

Theorem 3.2 Let k be a nonnegative integer with $k \leq m-1$. Let \mathcal{X} be a hypersurface of degree d over \mathbb{F}_q in \mathbb{P}^m . If $k_{\mathcal{X}} \leq k$, then

$$N_q(\mathcal{X}) \leq \theta_q(m-k-1) \cdot q^k(d-1) + \theta_q(k). \quad (7)$$

Furthermore, if $d \leq q$, the following conditions are equivalent:

- (a) Equality holds in (7);
- (b) $k_{\mathcal{X}} = k$, and for any \mathbb{F}_q -linear subspace \mathcal{L}_1 of dimension k with $\mathcal{L}_1 \subset \mathcal{X}$ and any \mathcal{M} of dimension $k+1$ with $\mathcal{L}_1 \subset \mathcal{M}$,

$$(*) \left\{ \begin{array}{l} \text{there are distinct } \mathbb{F}_q\text{-linear subspaces } \mathcal{L}_2, \dots, \mathcal{L}_d \text{ such that} \\ \mathcal{M} \cap \mathcal{X} = \mathcal{L}_1 \cup \dots \cup \mathcal{L}_d \text{ and } \cap_{i=1}^d \mathcal{L}_i \text{ is of dimension } k-1. \end{array} \right.$$

- (c) $k_{\mathcal{X}} = k$, and there is an \mathbb{F}_q -linear subspace \mathcal{L}_1 of dimension k with $\mathcal{L}_1 \subset \mathcal{X}$ such that for any \mathbb{F}_q -linear subspace \mathcal{M} of dimension $k+1$ with $\mathcal{L}_1 \subset \mathcal{M}$, the condition $(*)$ is fulfilled.

Proof. Put $\Phi(k, d) = \theta_q(m-k-1) \cdot q^k(d-1) + \theta_q(k)$.

Step 1. If $d \leq q+1$, then $\Phi(k+1, d) \geq \Phi(k, d)$. More precisely, if $d = q+1$, then $\Phi(k, q+1) = \theta_q(m)$ for any k ; and if $d \leq q$, then $\Phi(k+1, d) > \Phi(k, d)$.

Actually,

$$\begin{aligned} \Phi(k+1, d) - \Phi(k, d) &= \left(\theta_q(m-(k+1)-1) \cdot q - \theta_q(m-k-1) \right) q^k(d-1) + \theta_q(k+1) - \theta_q(k) \\ &= -q^k(d-1) + q^{k+1} = q^k((q+1)-d), \end{aligned}$$

which is nonnegative if $d \leq q+1$, and positive if $d < q+1$. It is obvious that $\Phi(k, q+1) = \theta_q(m)$.

Step 2. From Step 1, it is enough to show this theorem under the assumption $k_{\mathcal{X}} = k$. Choose any \mathbb{F}_q -linear subspace \mathcal{L}_1 of dimension k with $\mathcal{L}_1 \subset \mathcal{X}$. Let \mathbb{G} be the set of $(k+1)$ -dimensional \mathbb{F}_q -linear subspaces containing \mathcal{L}_1 . Each point P of $\mathcal{X} \setminus \mathcal{L}_1$ is contained in one and only one $(k+1)$ -dimensional \mathbb{F}_q -linear subspaces

$\mathcal{M} \in \mathbb{G}$, explicitly $\mathcal{M} = \langle \mathcal{L}_1, P \rangle$. Here $\langle \mathcal{L}_1, P \rangle$ denotes the linear subspace spanned by \mathcal{L}_1 and P . Hence

$$N_q(\mathcal{X}) = \sum_{\mathcal{M} \in \mathbb{G}} |(\mathcal{M} \cap \mathcal{X})(\mathbb{F}_q) \setminus \mathcal{L}_1(\mathbb{F}_q)| + N_q(\mathcal{L}_1).$$

Applying the lemma of Segre-Serre-Sørensen (2.2) for $\mathcal{M} \cap \mathcal{X} \subset \mathcal{M} = \mathbb{P}^{k+1}$,

$$N_q(\mathcal{M} \cap \mathcal{X}) \leq dq^k + \theta_q(k-1)$$

and when $d \leq q$ equality holds if and only if the condition $(*)$ is satisfied. On the other hand, \mathbb{G} forms the set of \mathbb{F}_q -points of projective space \mathbb{P}^{n-k-1} . Hence $|\mathbb{G}| = \theta_q(m-k-1)$ and

$$\begin{aligned} N_q(\mathcal{X}) &\leq \theta_q(m-k-1) \cdot (dq^k + \theta_q(k-1) - \theta_q(k)) + \theta_q(k) \\ &= \theta_q(m-k-1) \cdot q^k(d-1) + \theta_q(k) \end{aligned}$$

and when $d \leq q$ equality holds if and only if the condition $(*)$ is satisfied for all $\mathcal{M} \in \mathbb{G}$. This completes the proof. \square

Remark 3.3 If a hypersurface $\mathcal{X} \subset \mathbb{P}^m$ has no \mathbb{F}_q -hyperplane components, then $k_{\mathcal{X}} \leq m-2$. In this case, the bound (7) is just the elementary bound which we showed in [3].

Corollary 3.4 *Let \mathcal{X} be a nonsingular hypersurface of degree $d \geq 2$ of \mathbb{P}^m over \mathbb{F}_q .*

(i) *If m is odd, then*

$$N_q(\mathcal{X}) \leq \theta_q\left(\frac{m-1}{2}\right) \cdot ((d-1)q^{\frac{m-1}{2}} + 1).$$

(ii) *If m is even, then*

$$N_q(\mathcal{X}) \leq \theta_q\left(\frac{m}{2}\right)q^{\frac{m}{2}-1}(d-1) + \theta_q\left(\frac{m}{2} - 1\right).$$

Proof. If \mathcal{X} is nonsingular, $k_{\mathcal{X}} \leq \lfloor \frac{m-1}{2} \rfloor$ by Lemma 2.1. \square

4 Classification (the first step)

By Lemma 2.1, in order to show the main theorem (Theorem 1.1), it is enough to prove the following theorem.

Theorem 4.1 *Let n be an odd integer at least 3, and X a hypersurface of degree d of \mathbb{P}^n over \mathbb{F}_q . If $k_X \leq \frac{n-1}{2}$, then*

$$N_q(X) \leq \theta_q\left(\frac{n-1}{2}\right) \cdot ((d-1)q^{\frac{n-1}{2}} + 1). \quad (8)$$

Furthermore equality holds in (8) if and only if X is one of the hypersurfaces in the list described in Theorem 1.1.

The first part of this theorem has been already observed in Corollary 3.4.

First we get rid of the cases $d = 2$ and $d = q + 1$.

Proposition 4.2 *Let n be an odd integer at least 3, and X a quadratic hypersurface of \mathbb{P}^n over \mathbb{F}_q . If $k_X \leq \frac{n-1}{2}$ and $N_q(X) = \theta_q(\frac{n-1}{2})(q^{\frac{n-1}{2}} + 1)$, then X is the nonsingular hyperbolic quadric, that is, X is projectively equivalent over \mathbb{F}_q to the hypersurface*

$$\sum_{i=0}^{\frac{n-1}{2}} X_{2i} X_{2i+1} = 0.$$

Proof. For a general theory of quadrics over a finite field, consult [1, Chapter 5]. Since $k_X \leq \frac{n-1}{2} < n-1$, the quadric does not split into two hyperplanes over \mathbb{F}_q , that is, X is irreducible over \mathbb{F}_q . If X is not absolutely irreducible, then $X = H \cup H^{(q)}$ and $X(\mathbb{F}_q) = (H \cap H^{(q)})(\mathbb{F}_q)$, where H is a hyperplane over \mathbb{F}_{q^2} and $H^{(q)}$ is the q -Frobenius conjugate of H . This is a contradiction because $N_q(X) = \theta_q(\frac{n-1}{2})(q^{\frac{n-1}{2}} + 1) = \theta_q(n-1) + q^{\frac{n-1}{2}}$ and $N_q(H \cap H^{(q)}) = \theta_q(n-2)$. Therefore X is absolutely irreducible, and the possibilities of X are as follows:

(i) if X is nonsingular, then X is projectively equivalent over \mathbb{F}_q to either

$$\begin{aligned} \mathcal{H}_n : \sum_{i=0}^{\frac{n-1}{2}} X_{2i} X_{2i+1} &= 0; \text{ or} \\ \mathcal{E}_n : f(X_0, X_1) + \sum_{i=1}^{\frac{n-1}{2}} X_{2i} X_{2i+1} &= 0, \end{aligned}$$

where $f(X_0, X_1)$ is an irreducible quadratic polynomial over \mathbb{F}_q .

(ii) if X is a cone over a nonsingular quadric, then X is projectively equivalent over \mathbb{F}_q to either

$$\begin{aligned} \mathbb{P}^{n-2s-1} * \mathcal{P}_{2s} : X_0^2 + \sum_{i=1}^s X_{2i-1} X_{2i} &= 0 \text{ with } s \leq \frac{n-1}{2}; \text{ or} \\ \mathbb{P}^{n-2s} * \mathcal{H}_{2s-1} : \sum_{i=0}^{s-1} X_{2i} X_{2i+1} &= 0 \text{ with } s \leq \frac{n-1}{2}; \text{ or} \\ \mathbb{P}^{n-2s} * \mathcal{E}_{2s-1} : f(X_0, X_1) + \sum_{i=1}^{s-1} X_{2i} X_{2i+1} &= 0 \text{ with } s \leq \frac{n-1}{2}. \end{aligned}$$

If X is one of the following quadrics:

$$\begin{aligned} \mathbb{P}^{n-2s-1} * \mathcal{P}_{2s} &: \text{with } s \leq \frac{n-1}{2} - 1 \text{ or} \\ \mathbb{P}^{n-2s} * \mathcal{H}_{2s-1} &: \text{with } s \leq \frac{n-1}{2} \text{ or} \\ \mathbb{P}^{n-2s} * \mathcal{E}_{2s-1} &: \text{with } s \leq \frac{n-1}{2} - 1, \end{aligned}$$

then, $k_X > \frac{n-1}{2}$. Actually, $\mathbb{P}^{n-2s-1} * \mathcal{P}_{2s}$ contains the \mathbb{F}_q -linear subspace $X_0 = X_2 = X_4 = \dots = X_{2s} = 0$, which is of dimension $n - (s+1)$, bigger than $\frac{n-1}{2}$ if $s \leq \frac{n-1}{2} - 1$. $\mathbb{P}^{n-2s} * \mathcal{H}_{2s-1}$ contains $X_0 = X_2 = X_4 = \dots = X_{2(s-1)} = 0$, which is of dimension $n - s$, bigger than $\frac{n-1}{2}$ if $s \leq \frac{n-1}{2}$. $\mathbb{P}^{n-2s} * \mathcal{E}_{2s-1}$ contains $X_0 = X_1 = X_2 = X_4 = \dots = X_{2(s-1)} = 0$, which is of dimension $n - (s+1)$, bigger than $\frac{n-1}{2}$ if $s \leq \frac{n-1}{2} - 1$.

So the remaining possibilities are either \mathcal{H}_n or \mathcal{E}_n or $\mathbb{P}^0 * \mathcal{P}_{n-1}$ or $\mathbb{P}^1 * \mathcal{E}_{n-2}$. Since

$$\begin{aligned} N_q(\mathcal{H}_n) &= \theta_q\left(\frac{n-1}{2}\right)(q^{\frac{n-1}{2}} + 1) = \theta_q(n-1) + q^{\frac{n-1}{2}} \\ N_q(\mathcal{E}_n) &= \theta_q\left(\frac{n-3}{2}\right)(q^{\frac{n+1}{2}} + 1) = \theta_q(n-1) - q^{\frac{n-1}{2}} \\ N_q(\mathbb{P}^0 * \mathcal{P}_{n-1}) &= N_q(\mathcal{P}_{n-1})q + 1 = \theta_q(n-2)q + 1 = \theta_q(n-1) \\ N_q(\mathbb{P}^1 * \mathcal{E}_{n-2}) &= N_q(\mathcal{E}_{n-2})q^2 + \theta_q(1) = \theta_q(n-1) - q^{\frac{n+1}{2}}, \end{aligned}$$

X must be projectively equivalent over \mathbb{F}_q to \mathcal{H}_n . □

Proposition 4.3 *Let n be an odd integer at least 3, and X a hypersurface of degree $q+1$ of \mathbb{P}^n over \mathbb{F}_q . If $k_X \leq \frac{n-1}{2}$ and $N_q(X) = \theta_q(\frac{n-1}{2})(q^{\frac{n-1}{2}} + 1)$, then X is projectively equivalent over \mathbb{F}_q to the hypersurface*

$$\sum_{i=0}^{\frac{n-1}{2}} (X_{2i}^q X_{2i+1} - X_{2i} X_{2i+1}^q) = 0.$$

Proof. Since $\theta_q(\frac{n-1}{2})(q^{\frac{n-1}{2}} + 1) = \theta_q(n)$, $X(\mathbb{F}_q) = \mathbb{P}^n(\mathbb{F}_q)$. Hence the ideal of X is generated by $\{X_i^q X_j - X_i X_j^q \mid i < j\}$. Therefore, there is a q -alternating matrix A over \mathbb{F}_q such that X is given by the equation

$$(X_0^q, \dots, X_n^q) A \begin{pmatrix} X_0 \\ \vdots \\ X_n \end{pmatrix} = 0.$$

By the standard theory of alternating matrix over \mathbb{F}_q , we can choose new coordinates X_0, \dots, X_n of \mathbb{P}^n over \mathbb{F}_q so that A is of the form

$$\begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \\ & & & & & 0 \end{pmatrix},$$

that is, X is defined by $\sum_{i=0}^s (X_{2i}^q X_{2i+1} - X_{2i} X_{2i+1}^q) = 0$ with $s \leq \frac{n-1}{2}$. Obviously, $\{X_0 = X_2 = \dots = X_{2s} = 0\} \subset X$, and this \mathbb{F}_q -linear subspace is of dimension $n - (s + 1)$. Since $k_X \leq \frac{n-1}{2}$, we have $s = \frac{n-1}{2}$. \square

5 Classification (continuation)

To complete the proof of Theorem 4.1, we clarify the necessary set-up. In the previous section, two cases $d = 2$ and $q + 1$ were already handled.

Set-up 1 Let n be an odd integer at least 3, and X a hypersurface of \mathbb{P}^n over \mathbb{F}_q . Suppose that the degree d of X is in the range $2 < d \leq q$, $k_X = \frac{n-1}{2}$ and

$$N_q(X) = \theta_q\left(\frac{n-1}{2}\right) \cdot ((d-1)q^{\frac{n-1}{2}} + 1). \quad (9)$$

Note that initially the condition $k_X \leq \frac{n-1}{2}$ was supposed in Theorem 4.1, however, since we may assume that $d \leq q$ at this stage, the condition $k_X = \frac{n-1}{2}$ holds by Theorem 3.2.

Notation 5.1 The set of \mathbb{F}_q -linear subspaces of dimension u in \mathbb{P}^n is denoted by $G(u, \mathbb{P}^n)(\mathbb{F}_q)$.

Definition 5.2 For X in Set-up 1, $M \in G(\frac{n+1}{2}, \mathbb{P}^n)(\mathbb{F}_q)$ is said to be of type S (for X) if

$$M \cap X = L_1 \cup \dots \cup L_d,$$

where $L_1, \dots, L_d \in G(\frac{n-1}{2}, \mathbb{P}^n)(\mathbb{F}_q)$ and $\cap_{i=1}^d L_i \in G(\frac{n-3}{2}, \mathbb{P}^n)(\mathbb{F}_q)$. This $\frac{n-3}{2}$ -dimensional linear subspace is denoted by Λ_M .

The number of \mathbb{F}_q -points of $M \cap X$ above is given by:

Lemma 5.3

$$|(M \cap X)(\mathbb{F}_q)| = dq^{\frac{n-1}{2}} + \theta_q\left(\frac{n-3}{2}\right),$$

Proof. This is a direct consequence of Lemma 2.5. \square

Lemma 5.4 *Let $M \in G(\frac{n+1}{2}, \mathbb{P}^n)(\mathbb{F}_q)$. Then there is a linear space $L_1 \in G(\frac{n-1}{2}, \mathbb{P}^n)(\mathbb{F}_q)$ with $L_1 \subset X$ such that $L_1 \subset M$ if and only if M is of type S.*

Proof. The *if* part is obvious by definition. The *only if* part comes from Theorem 3.2, (a) \Rightarrow (b). \square

Remark 5.5 When $M \in G(\frac{n+1}{2}, \mathbb{P}^n)(\mathbb{F}_q)$ is of type S, then $\text{Sing}(M \cap X) = \Lambda_M$ by Lemma 2.4.

We need further notation:

Notation 5.6 $\bullet \mathbb{L} := \{L \in G(\frac{n-1}{2}, \mathbb{P}^n)(\mathbb{F}_q) \mid L \subset X\}.$

\bullet For $P \in X(\mathbb{F}_q)$, $\mathbb{L}(P) := \{L \in \mathbb{L} \mid L \ni P\}.$

Lemma 5.7 *For any $P \in X(\mathbb{F}_q)$, $\mathbb{L}(P) \neq \emptyset$.*

Proof. By Theorem 3.2, $\mathbb{L} \neq \emptyset$. Choose $L_1 \in \mathbb{L}$. Then either $P \in L_1$ or $P \notin L_1$. If the latter case occurs, then $M = \langle L_1, P \rangle \in G(\frac{n+1}{2}, \mathbb{P}^n)(\mathbb{F}_q)$. Then $M \cap X = L_1 \cup \dots \cup L_d$ by (5.4). Hence $P \in L_i$ for some i , that is, $L_i \in \mathbb{L}(P)$. \square

Lemma 5.8 *Let $L \in \mathbb{L}$. If $P \in X(\mathbb{F}_q) \setminus L$, then P is a nonsingular point of X .*

Proof. Let $M = \langle L, P \rangle \in G(\frac{n+1}{2}, \mathbb{P}^n)(\mathbb{F}_q)$, which is of type S by (5.4). Since $\text{Sing}(M \cap X) = \Lambda_M \subset L$ by (5.5), P is a nonsingular point of $M \cap X$. Hence so is P in X by (2.6). \square

Proposition 5.9 *Let P_0 be an \mathbb{F}_q -point of X . Suppose P_0 is a nonsingular point of X .*

- (i) *If $L_1 \in \mathbb{L}(P_0)$, then $L_1 \subset T_{P_0}X$, where $T_{P_0}X$ is the embedded tangent hyperplane to X at P_0 .*
- (ii) *Let $L_1 \in \mathbb{L}(P_0)$, and M of type S containing L_1 . If $M \subset T_{P_0}X$, then $P_0 \in \Lambda_M$.*
- (iii) *If M is of type S and $\Lambda_M \ni P_0$, then $M \subset T_{P_0}X$.*

Proof. (i) Since $P_0 \in L_1 \subset X$, we have $T_{P_0}L_1 = L_1$ (because L_1 itself is linear) and $T_{P_0}L_1 \subset T_{P_0}X$. Hence $L_1 \subset T_{P_0}X$.

(ii) Since M is of type S containing L_1 , there are $L_2, \dots, L_d \in \mathbb{L}$ such that $M \cap X = L_1 \cup L_2 \cup \dots \cup L_d$. Since P_0 is a singular point of $T_{P_0}X \cap X$ which is a hypersurface of $\mathbb{P}^{n-1} = T_{P_0}X$, it is also singular point of $(T_{P_0}X \cap X) \cap M$ by (2.6). Since $(T_{P_0}X \cap X) \cap M = X \cap M$ (because the assumption $M \subset T_{P_0}X$), $P_0 \in \text{Sing}(X \cap M) = \Lambda_M$.

(iii) There are \mathbb{F}_q -linear subspaces $L_1, \dots, L_d \in \mathbb{L}$ such that $M \cap X = L_1 \cup L_2 \cup \dots \cup L_d$ with $\Lambda_M = \cap_{i=1}^d L_i$. Hence $P_0 \in L_i$ for any $i = 1, \dots, d$. Since $L_i \subset T_{P_0}X$ by (i) and $\langle L_1, \dots, L_d \rangle = M$, we have $M \subset T_{P_0}X$. \square

Corollary 5.10 *Let $P_0, P_1 \in X(\mathbb{F}_q)$ be two distinct nonsingular points of X . Then $T_{P_0}X \ni P_1$ if and only if $T_{P_1}X \ni P_0$.*

Proof. Suppose the condition $T_{P_0}X \ni P_1$. We can find an \mathbb{F}_q -space $L_1 \in \mathbb{L}(P_0)$ by Lemma 5.7. When $P_1 \in L_1$, $L_1 = T_{P_1}L_1 \subset T_{P_1}X$. Since $P_0 \in L_1$, we have $P_0 \in T_{P_1}X$. When $P_1 \notin L_1$, let $M := \langle L_1, P_1 \rangle$. Since $L_1 = T_{P_0}L_1 \subset T_{P_0}X$ and $P_1 \in T_{P_0}X$ by the assumption, we have $M \subset T_{P_0}X$. Hence $P_0 \in \Lambda_M$ by (ii) of Proposition 5.9. Since $M \cap X = L_1 \cup L_2^{(M)} \cup \dots \cup L_d^{(M)}$ where $L_i^{(M)} \in \mathbb{L}$ ($i = 2, \dots, d$), there is an $L_i^{(M)}$ which contains P_1 . Hence $L_i^{(M)} \subset T_{P_1}X$. On the other hand, since $P_0 \in \Lambda_M \subset L_i^{(M)}$, we can conclude that $P_0 \in T_{P_1}X$. \square

Set-up 2 We keep Set-up 1. Additionally, fix a nonsingular point $P_0 \in X(\mathbb{F}_q)$ (the existence of such a point has been guaranteed by Lemma 5.8 and (9)), and also $L_1 \in \mathbb{L}(P_0)$. Let Y be the hypersurface $X \cap T_{P_0}X$ in $T_{P_0}X = \mathbb{P}^{n-1}$, which is also defined over \mathbb{F}_q and of degree d .

Lemma 5.11

$$N_q(Y) = \theta_q\left(\frac{n-3}{2}\right) \cdot q^{\frac{n-1}{2}}(d-1) + \theta_q\left(\frac{n-1}{2}\right).$$

Proof. Let

$$\mathbb{G} = \{M \in G\left(\frac{n+1}{2}, \mathbb{P}^n\right)(\mathbb{F}_q) \mid L_1 \subset M \subset T_{P_0}X\}.$$

Then \mathbb{G} forms a finite projective space $\mathbb{P}^{\frac{n-3}{2}}(\mathbb{F}_q)$. Obviously, $Y(\mathbb{F}_q) = \cup_{M \in \mathbb{G}}(M \cap X)(\mathbb{F}_q)$ and $M \cap M' = L_1$ if M and M' are distinct elements of \mathbb{G} . Hence

$$\begin{aligned} |Y(\mathbb{F}_q)| &= \sum_{M \in \mathbb{G}} (|(M \cap X)(\mathbb{F}_q)| - |L_1(\mathbb{F}_q)|) + |L_1(\mathbb{F}_q)| \\ &= \theta_q\left(\frac{n-3}{2}\right) q^{\frac{n-1}{2}}(d-1) + \theta_q\left(\frac{n-1}{2}\right), \end{aligned}$$

where the last equality comes from Lemma 5.3. \square

Set-up 3 We keep Set-ups 1 and 2. Furthermore, suppose $n \geq 5$. Take an \mathbb{F}_q -hyperplane $H \subset \mathbb{P}^n$ so that $H \not\ni P_0$. Then $T_{P_0}X \cap H$ is a linear subspace defined over \mathbb{F}_q of codimension 2 in \mathbb{P}^n . Let Z be the hypersurface

$$Y \cap (T_{P_0}X \cap H) \text{ in } T_{P_0}X \cap H = \mathbb{P}^{n-2},$$

which is also defined over \mathbb{F}_q and of degree d . Note that since $Y \subset T_{P_0}X$, Z is just a cutout of Y by H , that is, $Z = Y \cap H$.

Lemma 5.12

$$N_q(Z) = \theta_q\left(\frac{n-3}{2}\right) \cdot \left((d-1)q^{\frac{n-3}{2}} + 1\right).$$

Proof. Since

$$Y(\mathbb{F}_q) = \bigcup_{M \in \mathbb{G}} (L_1 \cup L_2^{(M)} \cup \dots \cup L_d^{(M)})(\mathbb{F}_q),$$

we have

$$Z(\mathbb{F}_q) = \bigcup_{M \in \mathbb{G}} \left((L_1 \cap H) \cup (L_2^{(M)} \cap H) \cup \dots \cup (L_d^{(M)} \cap H) \right)(\mathbb{F}_q).$$

Since $(M \cap H) \cap (M' \cap H) = L_1 \cap H$ if M and M' are distinct elements of \mathbb{G} ,

$$\begin{aligned} |Z(\mathbb{F}_q)| &= \sum_{M \in \mathbb{G}} \left(|((L_1 \cap H) \cup (L_2^{(M)} \cap H) \cup \dots \cup (L_d^{(M)} \cap H))(\mathbb{F}_q)| \right. \\ &\quad \left. - |(L_1 \cap H)(\mathbb{F}_q)| \right) + |(L_1 \cap H)(\mathbb{F}_q)|. \end{aligned} \quad (10)$$

For each $M \in \mathbb{G}$, since $\Lambda_M \ni P_0$ (5.9, ii) but $H \not\ni P_0$,

$$\dim L_1 \cap H = \dim L_2^{(M)} \cap H = \dots = \dim L_d^{(M)} \cap H = \frac{n-3}{2},$$

and

$$(L_1 \cap H) \cap (L_2^{(M)} \cap H) \cap \dots \cap (L_d^{(M)} \cap H) = \Lambda_M \cap H = \mathbb{P}^{\frac{n-5}{2}}.$$

Hence

$$|((L_1 \cap H) \cup (L_2^{(M)} \cap H) \cup \dots \cup (L_d^{(M)} \cap H))(\mathbb{F}_q)| = dq^{\frac{n-3}{2}} + \theta_q\left(\frac{n-5}{2}\right) \quad (11)$$

by Lemma 2.5. Therefore, by (10) and (11)

$$\begin{aligned} N_q(Z) &= \theta_q\left(\frac{n-3}{2}\right) \left(dq^{\frac{n-3}{2}} + \theta_q\left(\frac{n-5}{2}\right) - \theta_q\left(\frac{n-3}{2}\right) \right) + \theta_q\left(\frac{n-3}{2}\right) \\ &= \theta_q\left(\frac{n-3}{2}\right) \left((d-1)q^{\frac{n-3}{2}} + 1 \right). \quad \square \end{aligned}$$

Lemma 5.13

$$k_Z = \frac{n-3}{2}.$$

Proof. Since $L_1 \cap H \subset Z$, $k_Z \geq \frac{n-3}{2}$.

Suppose there is an $\frac{n-1}{2}$ -dimensional \mathbb{F}_q -linear space L_0 which is contained in $Z \subset X$. Then for each $Q \in Z(\mathbb{F}_q) \setminus L_0$, $M := \langle L_0, Q \rangle$ is of type S for X , and is contained in $T_{P_0}X \cap H = \mathbb{P}^{n-2}$ (because $L_0 \subset Z$ and $Q \in Z$). Let

$$\begin{aligned} \mathbb{G}' &:= \{M \in G\left(\frac{n+1}{2}, \mathbb{P}^n\right)(\mathbb{F}_q) \mid L_0 \subset M \subset \mathbb{P}^{n-2} = T_{P_0}X \cap H\} \\ &= \{M \in G\left(\frac{n+1}{2}, \mathbb{P}^{n-2}\right)(\mathbb{F}_q) \mid L_0 \subset M\} \\ &= \mathbb{P}^{\frac{n-5}{2}}(\mathbb{F}_q). \end{aligned}$$

Since

- (i) $Z(\mathbb{F}_q) = \cup_{M \in \mathbb{G}'} (M \cap X)(\mathbb{F}_q)$,
- (ii) $M \cap M' = L_0$ for distinct elements $M, M' \in \mathbb{G}'$ and
- (iii) $|(M \cap X)(\mathbb{F}_q)| = dq^{\frac{n-1}{2}} + \theta_q(\frac{n-3}{2})$ by Lemma 5.3,

we can compute the number of $Z(\mathbb{F}_q)$ as

$$\begin{aligned} Z(\mathbb{F}_q) &= \theta_q\left(\frac{n-5}{2}\right) \left(dq^{\frac{n-1}{2}} + \theta_q\left(\frac{n-3}{2}\right) - |L_0(\mathbb{F}_q)| \right) + |L_0(\mathbb{F}_q)| \\ &= \theta_q\left(\frac{n-5}{2}\right) (d-1)q^{\frac{n-1}{2}} + \theta_q\left(\frac{n-1}{2}\right). \end{aligned} \quad (12)$$

Compare this number (12) with that computed in Lemma 5.12. Namely,

$$\begin{aligned} &\left(\theta_q\left(\frac{n-5}{2}\right) (d-1)q^{\frac{n-1}{2}} + \theta_q\left(\frac{n-1}{2}\right) \right) - \left(\theta_q\left(\frac{n-3}{2}\right) ((d-1)q^{\frac{n-3}{2}} + 1) \right) \\ &= q^{\frac{n-3}{2}} (q+1-d), \end{aligned}$$

which is a contradiction because $d \leq q$. Therefore $k_Z = \frac{n-3}{2}$. \square

Theorem 5.14 *Under Set-up 1, q is square and $d = \sqrt{q} + 1$.*

Proof. When $n = 3$, we already know that the conclusion is true (Theorem 1.2). By Lemmas 5.12 and 5.13, the induction on odd n works well. \square

6 Classification for $d = \sqrt{q} + 1$

The remaining part of the classification is to determine the structure of X under Set-up 1 when $d = \sqrt{q} + 1$. Of course, throughout this section, q is supposed to be square.

When $n = 3$, we already know the surface X is a nonsingular Hermitian surface [5]. So we suppose that $n \geq 5$ as we did after Set-up 3. We keep the situation described in Set-ups 1 and 2.

Lemma 6.1 *The set $X(\mathbb{F}_q) \setminus T_{P_0}X$ is nonempty, and each point of this set is a nonsingular points of X .*

Proof. Note that $X(\mathbb{F}_q) \setminus T_{P_0}X = X(\mathbb{F}_q) \setminus Y(\mathbb{F}_q)$ because $Y = X \cap T_{P_0}X$ (see Set-up 2). By Set-up 1 and Lemma 5.11,

$$\begin{aligned} N_q(X) - N_q(Y) &= \\ &\theta_q\left(\frac{n-1}{2}\right) ((d-1)q^{\frac{n-1}{2}} + 1) - \left(\theta_q\left(\frac{n-3}{2}\right) q^{\frac{n-1}{2}} (d-1) + \theta_q\left(\frac{n-1}{2}\right) \right) \\ &= (d-1)q^{n-1} = q^{n-\frac{1}{2}} > 0. \end{aligned}$$

Hence $X(\mathbb{F}_q) \setminus T_{P_0}X \neq \emptyset$. Since $L_1 \in \mathbb{L}$ lies on $T_{P_0}X$ by Proposition 5.9 (i), any point of $X(\mathbb{F}_q) \setminus T_{P_0}X$ is nonsingular by Lemma 5.8. \square

Proposition 6.2 *Suppose n is an odd integer with $n \geq 5$. Let X be a hypersurface of degree $\sqrt{q} + 1$ in \mathbb{P}^n over \mathbb{F}_q with the conditions described in Set-up 1. Let Q_0 and Q_1 be points of $X(\mathbb{F}_q)$ that are nonsingular points of X with $T_{Q_1} \not\supset Q_0$. (Hence $T_{Q_0} \not\supset Q_1$ neither by Corollary 5.10.) Let $Y = X \cap T_{Q_0}X$, $Y' = X \cap T_{Q_1}X$, and*

$$Z = Y \cap T_{Q_1}X = Y' \cap T_{Q_0}X = X \cap T_{Q_0}X \cap T_{Q_1}X.$$

*Then $N_q(Z) = \theta_q(\frac{n-3}{2})(q^{\frac{n-2}{2}} + 1)$ and $k_Z = \frac{n-3}{2}$. Furthermore, $Y = Q_0 * Z$ in $T_{Q_0}X = \mathbb{P}^{n-1}$ and $Y' = Q_1 * Z$ in $T_{Q_1}X = \mathbb{P}^{n-1}$.*

Proof. Regard Q_0 as the point P_0 in Set-ups 2 and 3, and $T_{Q_1}X$ as the hyperplane H . Then $N_q(Z) = \theta_q(\frac{n-3}{2})(q^{\frac{n-2}{2}} + 1)$ by Lemma 5.12 with the assumption $d = \sqrt{q} + 1$, and also $k_Z = \frac{n-3}{2}$ by Lemma 5.13.

Choose coordinates X_1, \dots, X_n of $T_{Q_0}X = \mathbb{P}^{n-1}$ so that $Q_0 = (1, 0, \dots, 0)$ in \mathbb{P}^{n-1} and $T_{Q_0}X \cap T_{Q_1}X = \{X_1 = 0\}$ in \mathbb{P}^{n-1} . We want to apply the cone lemma (Proposition 2.8) to our situation, that is, regard the hypersurface Y of $\mathbb{P}^{n-1} = T_{Q_0}X$ as the hypersurface \mathcal{X} of \mathbb{P}^m in (2.8), $Z \subset \mathbb{P}^{n-2} = T_{Q_0}X \cap \{X_1 = 0\}$ as $\mathcal{Y} \subset \mathbb{P}^{k-1}$, and $Q_0 = \mathbb{P}^0$ as $\mathcal{L} = \mathbb{P}^{m-k}$. So m and k in the cone lemma are both $n - 1$ in the current situation. The first condition in (2.8) can be paraphrased in our situation as

$$N_q(Z) = \theta_q(\frac{n-3}{2})(q^{\frac{n-2}{2}} + 1) > \sqrt{q}q^{n-3} + \theta_q(n-4),$$

and it is not hard to check this inequality holds. The second condition in (2.8) obviously holds. To check the last condition, let $R \in Z(\mathbb{F}_q)$. Choose $L_1 \in \mathbb{L}(Q_0)$, and let $M = \langle L_1, R \rangle \subset T_{Q_0}X$ if $R \notin L_1$. Then $M \cap X = L_1^{(M)} \cup L_2^{(M)} \cup \dots \cup L_d^{(M)} \subset T_{Q_0}X$, and $Q_0 \in \Lambda_M = \cap_{i=1}^d L_i^{(M)}$, where $L_1^{(M)} = L_1$. Since there is an $L_i^{(M)}$ such that $R \in L_i^{(M)}$, the line $\langle Q_0, R \rangle$ is contained in $L_i^{(M)}$. Since $L_i^{(M)} \subset T_{Q_0}X \cap X = Y$, we can conclude that $(Q_0 * Z)(\mathbb{F}_q) \subset Y$.

Therefore, by the cone lemma, $Y = Q_0 * Z$. By the symmetry of the role of Q_0 and that of Q_1 , $Y' = Q_1 * Z$ also holds. \square

We finally prove the following theorem which completes the proof of Theorem 4.1.

Theorem 6.3 *Suppose n is an odd integer with $n \geq 3$. Let X be a hypersurface of degree $\sqrt{q} + 1$ in \mathbb{P}^n defined over \mathbb{F}_q . If $k_X = \frac{n-1}{2}$ and $N_q(X) = \theta_q(\frac{n-1}{2})(q^{\frac{n}{2}} + 1)$, then X is a nonsingular Hermitian hypersurface.*

Proof. When $n = 3$, this was already proved in [5]. So we assume that $n \geq 5$.

First we choose a point $P_0 \in X(\mathbb{F}_q)$ which fits with Set-ups 1 and 2. By Lemma 6.1, we can choose a point $P_1 \in X(\mathbb{F}_q) \setminus T_{P_0}X$, and it is a nonsingular point of X . Hence $P_0 \notin T_{P_1}X$ by Corollary 5.10. Choose coordinates X_0, X_1, \dots, X_n of \mathbb{P}^n over \mathbb{F}_q so that $P_0 = (0, 1, 0, \dots, 0)$, $P_1 = (1, 0, \dots, 0)$, $T_{P_0}X = \{X_0 = 0\}$ and $T_{P_1}X = \{X_1 = 0\}$. Note that if one applies a linear transformation of type

$$\begin{pmatrix} 1_2 & 0 \\ 0 & A \end{pmatrix} \quad (A \in GL(n-1, \mathbb{F}_q)),$$

to these coordinates, it does not affect the coordinate representations of P_0 and P_1 , and the equations of $T_{P_0}X$ and $T_{P_1}X$.

Let $Y = X \cap T_{P_0}X$, $Y' = X \cap T_{P_1}X$ and $Z = X \cap T_{P_0}X \cap T_{P_1}X$. Since $\mathbb{P}^{n-2} = T_{P_0}X \cap T_{P_1}X$ is defined by $X_0 = X_1 = 0$, we can regard X_2, \dots, X_n as coordinates of $T_{P_0}X \cap T_{P_1}X$. By Proposition 6.2, we can apply the induction hypothesis to Z , that is, Z is a nonsingular Hermitian hypersurface in $\mathbb{P}^{n-2} = T_{P_0}X \cap T_{P_1}X$. Therefore, we may assume that Z is defined by

$$\sum_{i=1}^{\frac{n-1}{2}} \left(X_{2i}^{\sqrt{q}} X_{2i+1} + X_{2i} X_{2i+1}^{\sqrt{q}} \right) = 0. \quad (13)$$

Since $Y = P_0 * Z$ and $Y' = P_1 * Z$ in $T_{P_0}X = \mathbb{P}^{n-1}$ and $T_{P_1}X = \mathbb{P}^{n-1}$ respectively, the equation (13) is also that for Y with coordinates X_0, X_2, \dots, X_n and that for Y' with coordinates X_1, \dots, X_n respectively. Therefore X is defined by $F = 0$ with

$$F = X_0 X_1 G(X_0, \dots, X_n) + \sum_{i=1}^{\frac{n-1}{2}} \left(X_{2i}^{\sqrt{q}} X_{2i+1} + X_{2i} X_{2i+1}^{\sqrt{q}} \right), \quad (14)$$

where $G(X_0, \dots, X_n)$ is a homogeneous polynomial of degree $\sqrt{q} - 1$. The partial derivations of F are as follows:

$$\begin{aligned} \frac{\partial F}{\partial X_0} &= X_1 G + X_0 X_1 \frac{\partial G}{\partial X_0} \\ \frac{\partial F}{\partial X_1} &= X_0 G + X_0 X_1 \frac{\partial G}{\partial X_1} \\ \frac{\partial F}{\partial X_{2i}} &= X_0 X_1 \frac{\partial G}{\partial X_{2i}} + X_{2i+1}^{\sqrt{q}} \quad (1 \leq i \leq \frac{n-1}{2}) \\ \frac{\partial F}{\partial X_{2i+1}} &= X_0 X_1 \frac{\partial G}{\partial X_{2i+1}} + X_{2i}^{\sqrt{q}} \quad (1 \leq i \leq \frac{n-1}{2}). \end{aligned} \quad (15)$$

For each $i = 1, 2, \dots, \frac{n-1}{2}$, let

$$\begin{aligned} P_{2i} &= (0, \dots, 0, \overset{2i}{0}, \overset{2i+1}{1}, 0, \dots, 0) \\ P_{2i+1} &= (0, \dots, 0, \overset{2i}{1}, \overset{2i+1}{0}, 0, \dots, 0). \end{aligned}$$

Then these points are nonsingular points of X , $T_{P_{2i}}X = \{X_{2i} = 0\}$, and $T_{P_{2i+1}}X = \{X_{2i+1} = 0\}$ by (15). Apply Proposition 6.2 to P_{2i} and P_{2i+1} . Then $X \cap T_{P_{2i}}X \cap T_{P_{2i+1}}X$ is also a nonsingular Hermitian hypersurface in $T_{P_{2i}}X \cap T_{P_{2i+1}}X = \mathbb{P}^{n-2}$ by the induction hypothesis.

Here we need a little more terminology: for letters X_0, \dots, X_n over \mathbb{F}_q , polynomials of type

$$X_k^{\sqrt{q}+1} \quad \text{or} \quad \lambda X_k^{\sqrt{q}} X_l + \lambda^{\sqrt{q}} X_k X_l^{\sqrt{q}} \quad (\lambda \in \mathbb{F}_q^\times)$$

are referred as Hermitian molecules. An equation of a Hermitian hypersurface, by definition, consists of an \mathbb{F}_q -linear combination of Hermitian molecules (but the converse is not true).

Since

$$F(X_0, \dots, X_{2i-1}, 0, \overset{2i}{0}, \overset{2i+1}{0}, X_{2i+1}, \dots, X_n) = 0 \quad (16)$$

is an equation of the Hermitian hypersurface $X \cap T_{P_{2i}}X \cap T_{P_{2i+1}}X$ in \mathbb{P}^{n-2} ,

$$X_0X_1G(X_0, \dots, X_{2i-1}, 0, \overset{2i}{0}, \overset{2i+1}{0}, X_{2i+1}, \dots, X_n)$$

consists of Hermitian molecules. Hence

$$G(X_0, \dots, X_{2i-1}, 0, \overset{2i}{0}, \overset{2i+1}{0}, X_{2i+1}, \dots, X_n) = c(\lambda X_0^{\sqrt{q}} + \lambda^{\sqrt{q}} X_1^{\sqrt{q}}) \quad (17)$$

for appropriate $\lambda \in \mathbb{F}_q$ and $c \in \overline{\mathbb{F}_q}$. Since the equation (16) defines a Hermitian hypersurface and the polynomial contains a pair of terms $X_{2j}^{\sqrt{q}}X_{2j+1} + X_{2j}X_{2j+1}^{\sqrt{q}}$ for some $j \geq 1$, we know $c \in \mathbb{F}_{\sqrt{q}}$, that is, we may assume c to be 1 in (17), and also

$$X_0X_1G(X_0, \dots, X_n) = X_0X_1(\lambda X_0^{\sqrt{q}} + \lambda^{\sqrt{q}} X_1^{\sqrt{q}}) + H(X_0, \dots, X_n) \quad (18)$$

with

$$H(X_0, \dots, X_{2i-1}, 0, \overset{2i}{0}, \overset{2i+1}{0}, X_{2i+1}, \dots, X_n) = 0. \quad (19)$$

We want to show $H(X_0, \dots, X_n)$ is the zero polynomial. Since the condition (19) holds for any i with $1 \leq i \leq \frac{n-1}{2}$ and X_0X_1 divides H , each monomial $X_0^{e_0}X_1^{e_1} \dots X_n^{e_n}$ appeared in H satisfies the condition

$$\begin{cases} e_0 + \dots + e_n = \sqrt{q} + 1 \\ e_0 > 0, \quad e_1 > 0 \\ e_{2i} + e_{2i+1} > 0 \quad \text{for } i \text{ with } 1 \leq i \leq \frac{n-1}{2}. \end{cases} \quad (20)$$

If $\sqrt{q} + 1 < 2 + \frac{n-1}{2}$, then no (e_0, e_1, \dots, e_n) satisfies (20). Hence, in this case, H is already the zero polynomial.

So we handle the opposit case below. Put

$$H(X_0, \dots, X_n) = \sum_{\mathbf{e}} c_{\mathbf{e}} X_0^{e_0} X_1^{e_1} \dots X_n^{e_n},$$

where $\mathbf{e} = (e_0, \dots, e_n)$ runs over the set of integer vectors satisfying (20).

Let ζ be a root of $t^{\sqrt{q}-1} = -1$, which is an element of \mathbb{F}_q . Take a pair of nonsingular points in $X(\mathbb{F}_q)$ such a way that

$$Q = (0, \dots, 0, 1, \overset{2i}{\zeta}, \overset{2i+1}{0}, 0, \dots, 0) \quad \text{and} \quad Q' = (0, \dots, 0, \zeta, \overset{2i}{1}, \overset{2i+1}{0}, 0, \dots, 0).$$

Since $\sqrt{q} + 1 \geq 2 + \frac{n-1}{2} \geq 4$, $\sqrt{q} - 1 \geq 2$. Also the characteristic of \mathbb{F}_q and $\sqrt{q} - 1$ are co-prime, we know $Q \neq Q'$.

Since $T_Q X = \{-\zeta X_{2i} + X_{2i+1} = 0\}$ and $T_{Q'} X = \{X_{2i} - \zeta X_{2i+1} = 0\}$, we can apply Proposition 6.2 to this situation. Especially, $X \cap T_Q X$ is a cone of a Hermitian hypersurface. Therefore

$$H(X_0, \dots, X_{2i}, \zeta^{2i+1} X_{2i}, X_{2i+2}, \dots, X_n)$$

consists of Hermitian molecules. Write down this polynomial explicitly:

$$\begin{aligned} H(X_0, \dots, X_{2i}, \zeta^{2i+1} X_{2i}, X_{2i+2}, \dots, X_n) \\ = \sum_{\mathbf{e}} c_{\mathbf{e}} \zeta^{e_{2i+1}} X_0^{e_0} \dots X_{2i-1}^{e_{2i-1}} X_{2i}^{e_{2i}+e_{2i+1}} X_{2i+2}^{e_{2i+2}} \dots X_n^{e_n} \\ = \sum_{\mathbf{e}'} \left(\sum_{v=0}^{\alpha} c_{(e_0, \dots, e_{2i-1}, \alpha-v, v, e_{2i+2}, \dots, e_n)} \zeta^v \right) X_0^{e_0} \dots X_{2i-1}^{e_{2i-1}} X_{2i}^{\alpha} X_{2i+2}^{e_{2i+2}} \dots X_n^{e_n}, \end{aligned}$$

where \mathbf{e}' is the abbreviation for a $(n-1)$ -pl $(e_0, \dots, e_{2i-1}, e_{2i+2}, \dots, e_n)$ in $(e_0, \dots, e_{2i-1}, \alpha-v, v, e_{2i+2}, \dots, e_n)$. Hence, for a fixed \mathbf{e}' ,

$$\sum_{v=0}^{\alpha} c_{(e_0, \dots, e_{2i-1}, \alpha-v, v, e_{2i+2}, \dots, e_n)} \zeta^v = 0 \quad (21)$$

for any $(\sqrt{q}-1)$ -root ζ of -1 . Since $\alpha \leq \sqrt{q} + 1 - (2 + \frac{n-3}{2}) < \sqrt{q} - 1$, all coefficients of ζ^v in (21) are 0. Hence H is the zero polynomial. Therefore

$$F = X_0 X_1 (\lambda X_0^{\sqrt{q}} + \lambda^{\sqrt{q}} X_1^{\sqrt{q}}) + \sum_{i=1}^{\frac{n-1}{2}} (X_{2i}^{\sqrt{q}} X_{2i+1} + X_{2i} X_{2i+1}^{\sqrt{q}}),$$

which means X is a Hermitian hypersurface. Since $P_0 = (0, 1, 0, \dots, 0)$ is a nonsingular point of X , $\lambda \neq 0$ by (15). Hence X is nonsingular. \square

7 Supplementary

In this section, we give two supplementaries.

7.1 Comparison with Koen Thas' bound

In [9], Thas already gave another bound for $N_q(\mathcal{X})$ involving the invariant $k_{\mathcal{X}}$, where \mathcal{X} is a hypersurface of \mathbb{P}^m of degree d over \mathbb{F}_q with $k_{\mathcal{X}} = k$. Suppose $1 \leq k \leq m-2$. Then he proved that

$$N_q(\mathcal{X}) \leq dq^{m-1} + \theta_q(m-2) + (d - (q+1)) \sum_{i=k}^{m-2} q^i \frac{\theta_q(m-1)}{\theta_q(i)\theta_q(i+1)}. \quad (22)$$

Proposition 7.1 *For d with $d \leq q+1$, the bound (7) is better than (22).*

Proof. Let S and T be the upper bounds in (7) and (22) respectively, namely,

$$S = \theta_q(m - k - 1) \cdot q^k(d - 1) + \theta_q(k)$$

and

$$T = dq^{m-1} + \theta_q(m - 2) + (d - (q + 1)) \sum_{i=k}^{m-2} q^i \frac{\theta_q(m - 1)}{\theta_q(i)\theta_q(i + 1)}.$$

The claim is $T - S > 0$ if $d \leq q + 1$ and $1 \leq k \leq m - 2$. It is easy to see that

$$S = \theta_q(m - 1) + q^k + (d - 2)q^k\theta_q(m - k - 1)$$

and

$$T = \theta_q(m - 1) + (d - 1)q^{m-1} + (d - (q + 1)) \sum_{i=k}^{m-2} q^i \frac{\theta_q(m - 1)}{\theta_q(i)\theta_q(i + 1)}.$$

Hence

$$T - S = q^{m-1} - (d - 2)q^k\theta_q(m - k - 2) - q^k + (d - (q + 1)) \sum_{i=k}^{m-2} q^i \frac{\theta_q(m - 1)}{\theta_q(i)\theta_q(i + 1)}. \quad (23)$$

Let $t = q + 1 - d$, which is nonnegative in the range of d . Then the second term of the right-hand side of (23) is rewritten as

$$-q^{k+1}\theta_q(m - k - 2) + (t + 1)q^k\theta_q(m - k - 2).$$

Hence

$$\begin{aligned} T - S = & q^{m-1} - q^{k+1}\theta_q(m - k - 2) + (t + 1)q^k\theta_q(m - k - 2) - q^k \\ & - t \sum_{i=k}^{m-2} q^i \frac{\theta_q(m - 1)}{\theta_q(i)\theta_q(i + 1)}. \end{aligned} \quad (24)$$

Furthermore, since

$$q^{m-1} - q^{k+1}\theta_q(m - k - 2) = -q^{k+1}\theta_q(m - k - 3)$$

and

$$-q^{k+1}\theta_q(m - k - 3) + q^k\theta_q(m - k - 2) - q^k = 0,$$

(24) becomes

$$T - S = t \left(q^k\theta_q(m - k - 2) - \sum_{i=k}^{m-2} q^i \frac{\theta_q(m - 1)}{\theta_q(i)\theta_q(i + 1)} \right).$$

Since

$$\frac{q^{i+1}}{\theta_q(i)\theta_q(i + 1)} = \frac{1}{\theta_q(i)} - \frac{1}{\theta_q(i + 1)},$$

we get

$$\begin{aligned}
T - S &= t \left(q^k \theta_q(m - k - 2) - \frac{\theta_q(m - 1)}{q} \left(\frac{1}{\theta_q(k)} - \frac{1}{\theta_q(m - 1)} \right) \right) \\
&= \frac{t}{q \theta_q(k)} (q^{k+1} \theta_q(m - k - 2) \theta_q(k) - \theta_q(m - 1) + \theta_q(k)) \\
&> \frac{t}{q \theta_q(k)} (q^{k+1} \theta_q(m - k - 2) - \theta_q(m - 1) + \theta_q(k)) = 0.
\end{aligned}$$

This completes the proof. \square

7.2 The case where m is even

In Corollary 3.4, we gave an upper bound for $N_q(\mathcal{X})$ even if \mathcal{X} is a nonsingular hypersurface in an even dimensional projective space \mathbb{P}^m . However, no nonsingular hypersurface achieves this upper bound if m is even. More precisely, we can say:

Annotation Suppose m is even. Let \mathcal{X} be a hypersurface of degree $d \geq 2$ of \mathbb{P}^m over \mathbb{F}_q with $k_{\mathcal{X}} \leq \frac{m}{2} - 1$. Then

$$N_q(\mathcal{X}) \leq \theta_q\left(\frac{m}{2}\right) q^{\frac{m}{2}-1} (d - 1) + \theta_q\left(\frac{m}{2} - 1\right),$$

however, equality no longer occurs.

Proof. This inequality comes from Theorem 3.2 like Corollary 3.4 (ii) did. Suppose equality holds for \mathcal{X} . Consider the ambient space \mathbb{P}^m as a hyperplane of \mathbb{P}^{m+1} , and take $P_0 \in \mathbb{P}^{m+1} \setminus \mathbb{P}^m$. Let $\tilde{\mathcal{X}} = P_0 * \mathcal{X}$ in \mathbb{P}^{m+1} . Then $\deg \tilde{\mathcal{X}} = \deg \mathcal{X}$, $k_{\tilde{\mathcal{X}}} = k_{\mathcal{X}} + 1$ and

$$\begin{aligned}
N_q(\tilde{\mathcal{X}}) &= N_q(\mathcal{X})q + 1 \\
&= \theta_q\left(\frac{m}{2}\right) q^{\frac{m}{2}} (d - 1) + \theta_q\left(\frac{m}{2} - 1\right) q + 1 \\
&= \theta_q\left(\frac{m}{2}\right) \left((d - 1) q^{\frac{m}{2}} + 1 \right).
\end{aligned}$$

Let $n = m + 1$. Then $\tilde{\mathcal{X}}$ satisfies the all assumptions of Theorem 4.1 and equality holds in (8). But from the latter part of this theorem, $\tilde{\mathcal{X}}$ must be nonsingular, which is a contradiction. \square

Finally we propose a conjecture for the case where m is even.

Conjecture Suppose $m (\geq 4)$ is an even integer. If \mathcal{X} is a nonsingular hypersurface of degree d in \mathbb{P}^m over \mathbb{F}_q . Then

$$N_q(\mathcal{X}) \leq \theta_q\left(\frac{m}{2} - 1\right) \left((d - 1) q^{\frac{m}{2}} + 1 \right)$$

might hold.

When $m = 2$, this inequality is just the Sziklai bound and holds with only one exception [2]. The nonsingular parabolic quadric hypersurface in \mathbb{P}^m , and the nonsingular Hermitian hypersurface in \mathbb{P}^m are examples for each of which equality holds.

References

- [1] J. W. P. Hirschfeld, *Projective geometries over finite fields* (Oxford mathematical monographs), Oxford University Press, New York, 1979.
- [2] M. Homma and S. J. Kim, *Sziklai's conjecture on the number of points of a plane curve over a finite field III*, Finite Fields Appl. 16 (2010) 315–319.
- [3] M. Homma and S. J. Kim, *An elementary bound for the number of points of a hypersurface over a finite field*, Finite Fields Appl. 20 (2013) 76–83.
- [4] M. Homma and S. J. Kim, *Numbers of points of surfaces in the projective 3-space over finite fields*, Finite Fields Appl. 35 (2015) 52–60.
- [5] M. Homma and S. J. Kim, *The characterization of Hermitian surfaces by the number of points*, J. Geom. 107 (2016) 509–521.
- [6] B. Segre, *Le geometrie di Galois*, Ann. Mat. Pura Appl. (4) 48 (1959) 1–96.
- [7] J.-P. Serre, *Lettre á M. Tsfasman*, Journées Arithmétiques, 1989 (Luminy, 1989), Astérisque 198-199-200 (1991), 351–353.
- [8] A. B. Sørensen, *On the number of rational points on codimension-1 algebraic sets in $\mathbf{P}^n(\mathbf{F}_q)$* , Discrete Math. 135 (1994), 321–334.
- [9] K. Thas, *On the number of points of a hypersurface in finite projective space (after J.-P. Serre)*, Ars Combin. 94 (2010), 183–190.
- [10] A. L. Tironi, *Hypersurfaces achieving the Homma-Kim bound*, preprint, 2014, available at arXiv: 1410.7320v2.